Criteria for linearized stability for a size-structured population model

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Abstract

We consider a size-structured aggregation and growth model of phytoplankton community proposed by Ackleh and Fitzpatrick [2]. The model accounts for basic biological phenomena in phytoplankton community such as growth, gravitational sedimentation, predation by zooplankton, fecundity, and aggregation. Our primary goal in this paper is to investigate the long-term behavior of the proposed aggregation and growth model. Particularly, using the well-known principle of linearized stability and semigroup compactness arguments, we provide sufficient conditions for local exponential asymptotic stability of zero solution as well as sufficient conditions for instability. We express these conditions in the form of an easy to compute characteristic function, which depends on the functional relationship between growth, sedimentation and fecundity. Our results can be used to predict long-term phytoplankton dynamics

Keywords: `Nonlinear evolution equations, principle of linearized stability, spectral analysis, structured populations dynamics, semigroup theory`

1. Introduction

Planktonic lifeforms provide a crucial source of food (organic carbon) for many aquatic species including blue whales [19]. In fact, oceanic plankton collectively provide approximately 40% of worlds organic carbon [8]. The main component of plankton community are unicellular algae called phytoplankton [16]. Similar to terrestrial plants, phytoplankton make their living by photosynthesis, and consequently inhabit surfaces of lakes and oceans. Phytoplankton cannot swim against a current and thus form aggregated communities at the surfaces of lakes and oceans to promote survival and proliferation. Besides their macroscopic predators such as whales and shrimp, aggregated phytoplankton community are also removed due to gravitational sedimentation and zooplankton grazing on phytoplankton.

To study the dynamic nature of phytoplankton communities various mathematical population models have been developed. The mathematical model that we consider in this article is the model first considered by Ackleh and Fitzpatrick in [2]. Although, the existence and uniqueness of a global positive solution in several different spaces have been proved in [2, 1, 4], the long-term behavior of this model has not been investigated. This is mainly due to the nonlinear nature of the Smoluchowski coagulation equations used for modeling aggregation. Hence, our main goal in this paper is to derive sufficient conditions for the local exponential asymptotic *stability* of the zero solution for the aggregation-growth model (see Section 2). In Section 3, we also derive sufficient conditions for *instability* of the zero solution. These conditions can then be used to predict long-term phytoplankton dynamics.

The size-structured population model proposed in [2] models growth of aggregates due to cell division and aggregation and removal of aggregates due to sedimentation and microscopic predation by zooplankton. In a phytoplankton community, the density of aggregates of size x at time t is denoted by $p(t, x)$. An aggregate is assumed to have minimum x_0 and maximum x_1 possible sizes. In vivo, there are no aggregates of volume 0 and the aggregates cannot grow indefinitely, so the only biologically plausible case is $0 < x_0 < x_1 < \infty$. Hence, in this paper we consider the case $0 < x_0 < x_1 < \infty$, and postpone the analysis of the case with $x_0 = 0$ and/or $x_1 = \infty$ for our future papers.

We will consider the following nonlinear partial integro-differential equation model for the evolution of a phytoplankton population,

$$\partial_t p(t, x) = -\partial_x(gp) + \mathcal{F}[p], \quad g(x_0)p(t, x_0) = \mathcal{K}[p](t), \quad p(0, x) = p_0(x) \in L^1[x_0, x_1] \tag{1}$$

where

$$\mathcal{F}[p](t, x) = \frac{1}{2}\int_{x_0}^{x-x_0} \beta(x-y, y)p(t, x-y)p(t, y)\, dy - p(t, x)\int_{x_0}^{x_1} \beta(x, y)p(t, y)\, dy - w(x)p(t, x)$$

and

$$\mathcal{K}[p](t) = \int_{x_0}^{x_1} q(x)p(t, x)dx\,. \tag{2}$$

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The function $g(x)$ represents the average growth rate of the aggregate of size x due to mitosis. Specifically, when a single cell in the aggregate of size x divides into two identical parent and daughter cells, the daughter cell enters the aggregate of size x contributing in a increase in total size. The coefficient $w(x)$ represents a size-dependent removal rate. Biologically, aggregates can be either removed by gravitational sedimentation or zooplankton grazing on phytoplankton [11]. $\beta(x,\, y)$ is the aggregation kernel, which describes the rate with which the aggregates of size x and y agglomerate to form an aggregate of size $x+y$. The fecundity rate $q(x)$ in (2) represents the number of new cells that fall off an aggregate of size x and enter single cell population.

As our solution space we use $H = L^1[x_0,\, x_1]$ with the usual norm $\|\cdot\|_{L^1}$ (hereafter, just $\|\cdot\|$). Consequently, the equation (1) can be written as a semilinear abstract Cauchy problem (ACP)

$$p_t = \mathcal{L}[p] + \mathcal{N}[p], \qquad p(0,\, x) = b_0(x) \in H\,. \tag{3}$$

The operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset H \to H$ is defined as

$$\mathcal{L}[p](x) = -\left(g(x)p(x)\right)' - w(x)p(x) \tag{4}$$

with its corresponding domain

$$\mathcal{D}(\mathcal{L}) = \{\phi \in H \,|\, (g\phi)' \in H,\, (g\phi)(x_0) = \mathcal{K}[\phi]\}\,. \tag{5}$$

The nonlinear operator $\mathcal{N} : H \to H$ is defined as

$$\mathcal{N}[p] = \frac{1}{2}\int_{x_0}^{x-x_0} \beta(x-y,\, y)p(x-y)p(y)\, dy - p(x)\int_{x_0}^{x_1} \beta(x,\, y)p(y)\, dy\,. \tag{6}$$

We make the following assumptions

$$g \in C^1[x_0,\, x_1]; \qquad \text{and } g(x) > 0 \text{ for } x_0 \le x \le x_1, \tag{$\mathscr{A}_g$}$$

$$\beta \in L^\infty([x_0,\, x_1]\times[x_0,\, x_1]); \qquad \beta(x,\, y) = \beta(y,\, x) \text{ and } \beta(x,\, y) = 0 \text{ if } x+y > x_1 \tag{$\mathscr{A}_\beta$}$$

$$w \in C[x_0,\, x_1] \qquad \text{and } w \ge 0 \text{ a.e. on } [x_0,\, x_1]\,, \tag{$\mathscr{A}_w$}$$

$$q \in L^\infty[x_0,\, x_1] \qquad \text{and } q \ge 0 \text{ a.e. on } [x_0,\, x_1]\,. \tag{$\mathscr{A}_q$}$$

Note that the restriction on g states that any aggregate of size $x \in [x_0,\, x_1]$ has strictly positive growth rate. Any aggregate growing out of the bounds are not considered in the model. The assumption on $w(x)$ enforces continuous dependence of the removal on the size of an aggregate and ensures that any aggregate size has a non-negative removal rate. Having the required ingredients in hand, we present the main result of this paper below, and demonstrate our proof in the subsequent sections.

Theorem 1. *Under the assumptions ($\mathscr{A}_g$), ($\mathscr{A}_\beta$), ($\mathscr{A}_w$) and ($\mathscr{A}_q$), the zero solution of the nonlinear evolution equation defined in* (1) *is locally asymptotically stable if*

$$\int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right)\, dx < 1\,.$$

Moreover, the zero solution is unstable if

$$\int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right)\, dx > 1\,.$$

2. Linearized stability for the zero solution

Our main goal in this section is to give sufficient conditions for the zero solution, $p(t,\, x) \equiv 0$, to be locally asymptotically stable. In particular, we use the linearized stability results introduced in [20]. For the convenience of readers we now present several results, which can be found in most of the semigroup theory books (see [7] for reference).

The growth bound, $\omega_0(\mathcal{A})$, of a strongly continuous semigroup $(T(t))_{t\ge 0}$ with an infinitesimal generator $\mathcal{A}$ is defined as

$$\omega_0(\mathcal{A}) := \inf\left\{\omega \in \mathbb{R} : \begin{array}{c} \exists M_\omega \ge 1 \text{ such that} \\ \|T(t)\| \le M_\omega e^{\omega t} \text{ for all } t \ge 0 \end{array}\right\}.$$

In the development below we denote $D\mathcal{A}(f)$ to be the Fréchet derivative of an operator $\mathcal{A}$ evaluated at f, which is defined as

$$D\mathcal{A}(u)h = \mathcal{A}[u+h] - \mathcal{A}[u] + o(h), \qquad \forall u \in \mathcal{D}(\mathcal{A})\,,$$

where o is little-o operator satisfying $\|o(h)\| \le b(r)\,\|h\|$ with increasing continuous function $b : [0,\, \infty) \to [0,\, \infty)$, $b(0) = 0$.

The stability results of this section are based on the following proposition from [20, p.198]. We refer readers to [14] for a generalized version of this proposition, which applies to a broader range of nonlinear evolution equations.

Proposition 2. *Let $(T(t))_{t\geq 0}$ be a C_0 semigroup in the Banach space X with an infinitesimal generator $\mathcal{L}$. Let $\mathcal{N} : X \to X$ be continuously Fréchet differentiable on X. Let $\bar{f} \in \mathcal{D}(\mathcal{L})$ be a stationary solution of (3), i.e., $(\mathcal{L}+\mathcal{N})[\bar{f}] = 0$. If the linearized operator $\mathcal{L} + D\mathcal{N}(\bar{f})$ (which is the infinitesimal generator of a C_0 semigroup by the well-known perturbation theorem) satisfies $\omega_0\left(\mathcal{L} + D\mathcal{N}(\bar{f})\right) < 0$, then $\bar{f}$ is locally asymptotically stable in the following sense:*

There exists η, $C \geq 1$, and $\alpha > 0$ such that if $\left\|f - \bar{f}\right\| < \eta$, then a unique mild solution $u(t)$ of (3),

$$u(t) = T(t)f + \int_0^t T(t-s)\mathcal{N}[u(s)]\, ds\,,$$

exists for all $t \geq 0$ and $\left\|u(t) - \bar{f}\right\| \leq Ce^{-\alpha t}\left\|f - \bar{f}\right\|$ for all $t \geq 0$.

In [4] authors have proved that the linear operator $\mathcal{L}$ generates a strongly continuous semigroup. Hence, we prove the second assumption of Proposition 2 below.

Lemma 3. *The nonlinear operator $\mathcal{N}$ defined in* (6) *is continuously Fréchet differentiable on H.*

Proof. The Fréchet derivative of the nonlinear operator $\mathcal{N}$ is given explicitly as

$$D\mathcal{N}(\phi)[h(x)] = \frac{1}{2}\int_{x_0}^{x-x_0} \beta(x-y,\, y)\,[\phi(y)h(x-y) + h(y)\phi(x-y)]\, dy - h(x)\int_{x_0}^{x_1} \beta(x,\, y)\phi(y)dy - \phi(x)\int_{x_0}^{x_1} \beta(x,\, y)h(y)dy\,.$$

For the arbitrary functions u_1, $u_2 \in H$ we have

$$\begin{aligned}|D\mathcal{N}(u_1)h(x) - D\mathcal{N}(u_2)h(x)| \leq{}& \frac{1}{2}\left\|\beta\right\|_\infty \int_{x_0}^{x_1} |u_1(y) - u_2(y)||h(x-y)|\, dy + \frac{1}{2}\left\|\beta\right\|_\infty \int_{x_0}^{x_1} |h(y)||u_1(x-y) - u_2(x-y)|\, dy \\ &+ |h(x)|\left\|\beta\right\|_\infty \int_{x_0}^{x_1} |u_1(y) - u_2(y)|dy + |u_1(x) - u_2(x)|\left\|\beta\right\|_\infty \int_{x_0}^{x_1} |h(y)|dy\end{aligned}$$

An application of Young's inequality for convolutions (see [3, Theorem 2.24]) to the first two integrals gives,

$$|D\mathcal{N}(u_1)h(x) - D\mathcal{N}(u_2)h(x)| \leq \left\|\beta\right\|_\infty \left\|u_1 - u_2\right\| \left\|h\right\| + |h(x)|\left\|\beta\right\|_\infty \left\|u_1 - u_2\right\| + |u_1(x) - u_2(x)|\left\|\beta\right\|_\infty \left\|h\right\|\,.$$

Consequently, taking the integral of both sides with respect to x yields

$$\left\|D\mathcal{N}(u_1)h(x) - D\mathcal{N}(u_2)h(x)\right\| \leq (x_1 - x_0)\left\|\beta\right\|_\infty \left\|u_1 - u_2\right\| \left\|h\right\| + \left\|\beta\right\|_\infty \left\|u_1 - u_2\right\| \left\|h\right\| + \left\|u_1 - u_2\right\| \left\|\beta\right\|_\infty \left\|h\right\|$$

for all $h \in H$. Then it follows that

$$\left\|D\mathcal{N}(u_1) - D\mathcal{N}(u_2)\right\| \leq (x_1 - x_0 + 2)\left\|\beta\right\|_\infty \left\|u_1 - u_2\right\|\,,$$

which in turn implies that the nonlinear operator $\mathcal{N}$ is continuously Fréchet differentiable on H. ☐

For the zero solution, $\bar{f} = 0$, we have $D\mathcal{N}(\bar{f}) = 0$. Proposition 2 implies that in order to derive a sufficient condition, we also have to show that $\omega_0(\mathcal{L}) < 0$. To achieve that we follow the steps introduced in [7, §6.1] and [10]. First, we show that C_0 semigroup $(T(t))_{t\geq 0}$ generated by $\mathcal{L}$ is, in fact, a positive semigroup on the *Banach lattice* H with the usual sense of ordering "$\geq$" and the absolute value, $|\cdot|$.[1]

Proposition 4. *The linear operator $\mathcal{L}$ on $H = L^1[x_0,\, x_1]$ generates a positive C_0 semigroup $(T(t))_{t\geq 0}$. Hence, the growth bound $\omega_0(\mathcal{L})$ is equal to the spectral bound $s(\mathcal{L})$, i.e., $s(\mathcal{L}) = \omega_0(\mathcal{L})$.*

Proof. For given $\lambda \in \mathbb{C}$, the resolvent operator of $\mathcal{L}$ is given explicitly by

$$R(\lambda,\, \mathcal{L})\phi = \phi(x_0)g(x_0)\frac{q(x)}{g(x)}\exp\left(-\int_{x_0}^{x} \frac{\lambda + w(s)}{g(s)}\, ds\right) + \frac{1}{g(x)}\int_{x_0}^{x} \phi(y)\exp\left(-\int_{y}^{x} \frac{\lambda + w(s)}{g(s)}\, ds\right) dy \tag{7}$$

Using the similar reasoning to Lemma 2.4 of [4] we conclude that for all $\lambda \in \mathbb{C}$ with $Re(\lambda) > \left\|g\right\|_\infty + \left\|q\right\|_\infty$, the resolvent operator $R(\lambda,\, \mathcal{L})$ exists. Thus, the resolvent set of $\mathcal{L}$ is not empty, i.e., $\rho(\mathcal{L}) \not\equiv \emptyset$. Then, since g, w, and q are positive functions on $[x_0,\, x_1]$, it is straightforward to see that $R(\lambda,\, \mathcal{L})$ is a positive operator whenever it exists. Consequently, the positivity of $T(t)$ follows from the Characterization Theorem in [7, p.353]. Since $H = L^1[x_0,\, x_1]$ is a Banach lattice and $(T(t))_{t\geq 0}$ is a positive semigroup, the last statement of the proposition follows from the main theorem presented in [21]. ☐

[1] We refer readers to [18] for a detailed definition of a positive semigroup on a Banach lattice.

Recall that an operator $\mathcal{A}$ has a *compact resolvent* if $\rho(\mathcal{A}) \neq \emptyset$ and $R(\lambda, \mathcal{A})$ is *compact* for all $\lambda \in \rho(\mathcal{A})$. Moreover, if one can prove that $R(\lambda, \mathcal{A})$ is compact for some $\lambda \in \rho(\mathcal{A})$, then $R(\lambda, \mathcal{A})$ is compact for all $\lambda \in \rho(\mathcal{A})$. Operators which have a compact resolvent have the very nice property that their spectrum $\sigma(\mathcal{A})$ coincide with their point spectrum $\sigma_p(\mathcal{A})$, i.e.,

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) := \{\lambda \in \mathbb{C} \,|\, \mathcal{A}\phi = \lambda\phi \text{ for some } \phi \neq 0 \in X\} \,, \tag{8}$$

where X is some Banach space and $\mathcal{A}$ is an operator on X with a compact resolvent.

Before proceeding further we first prove the following lemma, which we use in the proof of the next proposition.

Lemma 5. *For $\phi \in H$ and $\lambda_1 \in \mathbb{C}$ with $Re(\lambda_1) > \|g\|_\infty + \|q\|_\infty$ we define the linear operators*

$$K_1[\phi](x) = \phi(x_0)g(x_0) = \int_{x_0}^{x_1} q(y)\phi(y)\, dy$$

and

$$K_2[\phi](x) = \int_{x_0}^{x} \phi(y) \exp\left(-\int_y^x \frac{\lambda_1 + w(s)}{g(s)}\, ds\right) dy \,.$$

Then K_1 and K_2 are compact on H.

Proof. We first prove that K_2 is compact on H. Recall that an operator is compact if it maps bounded sets into relatively compact sets. In order to show that a set is relatively compact in L^1, we use the Kolmogorov-Riesz compactness theorem (see *[12]* for a statement of the theorem).

Define a unit ball in $H = L^1[x_0,\, x_1]$ as $B = \{\phi \in H \,|\, \|\phi\| \leq 1\}$. Now, we prove that $K_2 B$ is relatively compact by proving that each condition of the Kolmogorov-Riesz theorem applies to the set $K_2 B$. For a given function $\phi \in B$,

$$\begin{aligned} \|K_2\phi(x)\| &= \int_{x_0}^{x_1} \left| \int_{x_0}^{x} \phi(y) \exp\left(-\int_y^x \frac{\lambda_1 + w(s)}{g(s)}\, ds\right) dy \right| dx \leq \int_{x_0}^{x_1}\int_{x_0}^{x} |\phi(y)|\, dy\, dx \\ &\leq (x_1 - x_0)\, \|\phi\| \leq (x_1 - x_0)\,. \end{aligned}$$

This implies that the set $K_2 B$ is bounded. So the first condition of the Kolmogorov-Riesz theorem is satisfied. Since we are working on a finite domain $[x_0,\, x_1]$, the second condition of the Kolmogorov-Riesz theorem is satisfied by default.

For convenience, let us define

$$k(x,\, y) := \exp\left(-\int_y^x \frac{\lambda_1 + w(s)}{g(s)}\, ds\right).$$

Observe that for $\lambda_1 \in \mathbb{C}$ with $Re(\lambda_1) > \|g\|_\infty + \|q\|_\infty$, $k(x,\, y)$ is uniformly continuous on $[x_0,\, x_1] \times [x_0,\, x_1]$. So for a given $\varepsilon_1 > 0$ there exist $\delta_1 > 0$ such that

$$|k(x+h,\, y) - k(x,\, y)| < \varepsilon_1 \text{ for all } |h| < \delta_1 \text{ and } y \in [x_0,\, x_1]\,. \tag{9}$$

Furthermore, notice that $|k(x,\, y)| \leq 1$ for $x \geq y$. Consequently, for a given $\varepsilon > 0$ and $\phi \in B$ we have

$$\begin{aligned} \int_{x_0}^{x_1} \left| \int_{x_0}^{x+h} k(x+h,\, y)\phi(y)\, dy - \int_{x_0}^{x} k(x,\, y)\phi(y)\, dy \right| dx &\leq \int_{x_0}^{x_1} \left| \int_{x}^{x+h} k(x+h,\, y)\phi(y)\, dy \right| dx \\ &\quad + \int_{x_0}^{x_1} \left| \int_{x_0}^{x} \left[k(x+h,\, y) - k(x,\, y)\right] \phi(y)\, dy \right| dx \\ &\leq \int_{x_0}^{x_1} \left| \int_{x}^{x+h} |k(x+h,\, y)| \cdot \|\phi\|\, dy \right| dx + \int_{x_0}^{x_1}\int_{x_0}^{x} \varepsilon_1 |\phi(y)|\, dy\, dx \\ &\leq (x_1 - x_0)|h| \cdot \|\phi\| + \varepsilon_1 (x_1 - x_0) \cdot \|\phi\| \\ &\leq (\delta_1 + \varepsilon_1)(x_1 - x_0) < \varepsilon \end{aligned}$$

for sufficiently small ε_1 and δ_1 in (9). This proves the third condition, and thus from the Kolmogorov-Riesz theorem the set $K_2 B$ is relatively compact in H. This in turn implies that K_2 is a compact operator on H.

For the operator K_1 observe that the set $K_1 B$ is bounded, i.e.,

$$\|K_1\phi(x)\| = \int_{x_0}^{x_1} \left| \int_{x_0}^{x_1} \phi(y)q(y)\, dy \right| dx \leq (x_1 - x_0)\, \|q\|_\infty \, \|\phi\| \leq (x_1 - x_0)\, \|q\|_\infty \,.$$

Since the function $q(y)$, inside the integral, does not depend on x, the third condition of the Kolmogorov-Riesz theorem follows immediately. Hence, the operator K_1 is also compact. $\square$

Proposition 6. *The operator $\mathcal{L}$ defined in* (4) *has a compact resolvent.*

Proof. From Proposition 4 we already know that the resolvent set of $\mathcal{L}$ is not empty. So we only have to prove that $R(\lambda, \mathcal{L})$ is compact for some $\lambda \in \rho(\mathcal{L})$. Particularly, we will prove that $R(\lambda_1, \mathcal{L})$ is compact for any $\lambda_1 \in \mathbb{C}$ with $Re(\lambda_1) > \|g\|_\infty + \|q\|_\infty$. The resolvent operator defined in (7) can be written as the sum of the compositions of linear operators

$$R(\lambda_1, \mathcal{L}) = B_1 K_1 + B_2 K_2 \,,$$

where K_1 and K_2 are defined as in Lemma 5,

$$B_1[\phi](x) = \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{\lambda_1 + w(s)}{g(s)}\, ds\right) \phi(x) \,,$$

and

$$B_2[\phi](x) = \frac{1}{g(x)} \phi(x) \,.$$

From the assumptions ($\mathscr{A}_q$) and ($\mathscr{A}_g$) it follows that the operators B_1 and B_2 are bounded. In Lemma 5 we have proved that the operators K_1 and K_2 are compact on H. Then the operator $B_1 K_1$, composition of a bounded and a compact operator, is compact (see Proposition 5.43 of [13]). Similarly, the operator $B_2 K_2$ is compact. This in turn implies that the resolvent operator $R(\lambda_1, \mathcal{L})$, a linear combination of compact operators, is also compact. □

The above proposition together with equation (8) implies that the spectrum of the operator $\mathcal{L}$ consists of only eigenvalues, i.e., $\sigma(\mathcal{L}) = \sigma_p(\mathcal{L})$. Thus, we can now characterize the spectrum of $\mathcal{L}$ by its eigenvalues.

Proposition 7. *For $\lambda \in \mathbb{C}$,*

$$\lambda \in \sigma_p(\mathcal{L}) = \sigma(\mathcal{L}) \;\Leftrightarrow\; \xi(\lambda) = 0 \,,$$

where

$$\xi(\lambda) = \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{\lambda + w(s)}{g(s)}\, ds\right) dx - 1$$

is a characteristic function of $\mathcal{L}$.

Proof. The eigenvalue equation for the operator $\mathcal{L}$ is given by

$$\mathcal{L}\phi - \lambda\phi = -\left(g(x)\phi(x)\right)' - w(x)\phi(x) - \lambda\phi = 0 \,.$$

The solution of the above equation is given by the following eigenfunction

$$\phi(x) = \frac{\phi(x_0) g(x_0)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{\lambda + w(s)}{g(s)}\, ds\right) . \tag{10}$$

We note that

$$\begin{aligned}
\|\phi\| &= \int_{x_0}^{x_1} \frac{|\phi(x_0) g(x_0)|}{g(x)} \exp\left(-Re(\lambda) \int_{x_0}^{x} \frac{1}{g(s)}\, ds\right) \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right) dx \\
&\leq \int_{x_0}^{x_1} \frac{|\phi(x_0) g(x_0)|}{g(x)} \exp\left(-Re(\lambda) \int_{x_0}^{x} \frac{1}{g(s)}\, ds\right) dx \\
&\leq (x_1 - x_0) |\phi(x_0) g(x_0)| \left\| \frac{1}{g(x)} \right\|_\infty \exp\left(-Re(\lambda) \int_{x_0}^{x_1} \frac{1}{g(s)}\, ds\right) ,
\end{aligned}$$

and hence $\phi \in H$. From assumption ($\mathscr{A}_g$) we have $g \in C^1[x_0, x_1]$. This in turn implies that $g'\phi \in H$. Analogously, we can also prove that $(g\phi)' = g'\phi + g\phi' \in H$.

Lastly, in order for $\phi \in \mathcal{D}(\mathcal{L})$ we should have

$$g(x_0)\phi(x_0) = \mathcal{K}[\phi] = \int_{x_0}^{x_1} \phi(x_0) g(x_0) \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{\lambda + w(s)}{g(s)}\, ds\right) ,$$

which is equivalent to

$$0 = \xi(\lambda) = \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{\lambda + w(s)}{g(s)}\, ds\right) dx - 1$$

□

Since the eigenfunction ϕ defined in (10) is L^1, the characteristic function $\xi(\lambda)$ is a finite-valued function for all $\lambda \in \mathbb{C}$. Moreover, when $\xi(\lambda)$ is restricted to $\mathbb{R}$ it is strictly decreasing. Furthermore, a simple limit calculation shows that

$$\lim_{\lambda \to \infty} \xi(\lambda) = -1 \text{ and } \lim_{\lambda \to -\infty} \xi(\lambda) = \infty \,.$$

This in turn, from the Intermediate Value Theorem, implies that there exists a unique $\lambda_0 \in \mathbb{R}$ such that $\xi(\lambda_0) = 0$. We can also guarantee that this eigenvalue λ_0 is negative real number provided that we have $\xi(0) < 0$.

Remark 8. We also claim that the spectral bound of $\mathcal{L}$ is equal to this λ_0, i.e., $s(\mathcal{L}) = \lambda_0$. Suppose that there exists $\lambda_1 \in \sigma(\mathcal{L}) = \sigma_p(\mathcal{L}) \subseteq \mathbb{C}$ such that $Re(\lambda_1) > \lambda_0$. On the other hand, from Proposition 7, λ_1 should be a zero of characteristic function. i.e. $\xi(\lambda_1) = 0$. However,

$$\begin{aligned}
1 = Re(\xi(\lambda_1)) + 1 &= \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \cos\left[\int_{x_0}^{x} \frac{Im(\lambda_1)}{g(s)}\, ds\right] \exp\left(-\int_{x_0}^{x} \frac{Re(\lambda_1) + w(s)}{g(s)}\, ds\right) dx \\
&\le \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{Re(\lambda_1) + w(s)}{g(s)}\, ds\right) dx \\
&< \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{\lambda_0 + w(s)}{g(s)}\, ds\right) dx \\
&= 1\,,
\end{aligned}$$

which is a contradiction.

We summarize the discussion of this section in the following criterion.

Criterion. *(Stability) The spectral bound of $\mathcal{L}$ is a unique real number λ_0 such that $\xi(\lambda_0) = 0$ and hence $\omega_0(\mathcal{L}) = s(\mathcal{L}) = \lambda_0$. Moreover, the zero solution of the semilinear evolution equation* (3) *is locally exponentially stable if*

$$\xi(0) = \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right) - 1 < 0\,.$$

3. Linearized instability for the zero solution

In this section we derive sufficient conditions for instability of the zero solution. In particular, we use the following proposition from [20, p.206].

Proposition 9. *Let $(T(t))_{t\ge 0}$ be a C_0 semigroup in the Banach space X with infinitesimal generator $\mathcal{L}$. Let $\mathcal{N} : X \to X$ be continuously Fréchet differentiable on X. Let $\bar{f} \in \mathcal{D}(\mathcal{L})$ be a stationary solution of* (3). *If there exists $\lambda_0 \in \sigma(\mathcal{L} + D\mathcal{N}(\bar{f}))$ such that $Re(\lambda_0) > 0$ and*

$$\max\left\{\omega_1(\mathcal{L} + D\mathcal{N}(\bar{f})), \sup_{\lambda \in \sigma_D(\mathcal{L}+D\mathcal{N}(\bar{f}))\setminus\{\lambda_0\}} Re(\lambda)\right\} < Re(\lambda_0)\,, \tag{11}$$

then $\bar{f}$ is an unstable equilibrium in the sense that there exists $\varepsilon > 0$ and sequence $\{f_n\}$ in X such that $f_n \to \bar{f}$ and $\left\|T(n)f_n - \bar{f}\right\| \ge \varepsilon$ for $n = 1, 2, \dots$.

The *discrete spectrum* of an operator $\mathcal{A}$ on a Banach space X, denoted by $\sigma_D(\mathcal{A})$, is the subset of $\lambda \in \sigma_p(\mathcal{A})$ such that λ is an isolated eigenvalue of finite multiplicity, i.e., the dimension of the set $\{\psi \in X : \mathcal{A}\psi = \lambda\psi\}$ is finite and nonzero. Let $(T(t))_{t\ge 0}$ be a C_0 semigroup on the Banach space X with its generator $\mathcal{A}$. Then the limit $\omega_1(\mathcal{A}) = \lim_{t\to\infty} t^{-1} \log\left(\alpha[T(t)]\right)$ is called α-growth bound of $(T(t))_{t\ge 0}$. Here, $\alpha[T(t)]$ is a measure of non-compactness of $T(t)$. The measure of non-compactness, introduced in a textbook [15], associates numbers to operators (or sets), which tells how close is an operator (or a set) to a compact operator (or set). For instance, $\alpha[T(t)] = 0$ implies that the semigroup $(T(t))_{t\ge 0}$ is eventually compact. In general, computing an explicit value of the $\alpha-$growth bound ω_1 is a complicated task. However, if we can prove that the linear operator $\mathcal{L} + D\mathcal{N}(\bar{f})$ generates an eventually compact C_0 semigroup, then from [20, Remark 4.8] it follows that $\omega_1(\mathcal{L} + D\mathcal{N}(\bar{f})) = -\infty$.

Proposition 10. *The semigroup $T(t)$ generated by the operator $\mathcal{L}$ is eventually compact. Specifically, it is compact for $t > 2\Gamma(x_1)$, where*

$$\Gamma(x) := \int_{x_0}^{x} \frac{1}{g(s)}\, ds\,.$$

Proof. We first show that the semigroup $(T(t))_{t\geq 0}$ is differentiable for $t > 2\Gamma(x_1)$. Note that eventual differentiability implies eventual norm continuity (see Diagram 4.26 in [7, p.119]). Consequently, the result follows from the compactness of the resolvent set of $\mathcal{L}$ (Proposition 6) and [7, Lemma 4.28].

The following proof has been adopted from the proof of [9, Theorem 3.1]. The abstract Cauchy problem $u_t = \mathcal{L}u$ can be rewritten as a partial differential equation

$$u_t(t,\, x) + g(x)u_x(t,\, x) + \left(g'(x) + w(x)\right) u(t,\, x) = 0\,. \tag{12}$$

By Thoerem 2.2 in [5] we know that the semigroup $(T(t))_{t\geq 0}$ generated by $\mathcal{L}$ is given explicitly by

$$T(t)\varphi(x) = \begin{cases} \varphi\left(\Gamma^{-1}\left(\Gamma(x) - t\right)\right) \frac{g\left(\Gamma^{-1}(\Gamma(x)-t)\right)}{g(x)} \exp\left(-\int_{\Gamma^{-1}(\Gamma(x)-t)}^{x} \frac{w(s)}{g(s)}\, ds\right) & t \leq \Gamma(x) \\ \frac{1}{g(x)} \mathcal{K}\left(T\left(t - \Gamma(x)\right)\varphi(x)\right) \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right) & \Gamma(x) < t \end{cases}. \tag{13}$$

Thus for $t > \Gamma(x)$, we have

$$u(t,\, x) = \frac{1}{g(x)} \int_{x_0}^{x_1} q(x)u(t - \Gamma(x),\, x)dx \times \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right)\,.$$

On the other hand, since $g \in C^1[x_0,\, x_1]$ and $g(x) > 0$ we have $\Gamma(x) \in C[x_0, x_1]$. Therefore, $u(t,\, x)$ is continuous both in x and t for $t > \Gamma(x_1)$, where $\Gamma(x_1)$ is the maximum of $\Gamma(x)$ on $[x_0,\, x_1]$. Then, from (12), $u(t,\, x)$ is continuously differentiable for $t > 2\Gamma(x_1)$. □

In Proposition 6 we have proved that the spectrum of $\mathcal{L}$ consists of only eigenvalues, which can be expressed as the zeros of the characteristic equation $\xi(\lambda)$. Consequently, similar to Remark 8, we can show that there exist a unique real number $\lambda_0 > 0$ such that

$$\sup_{\lambda \in \sigma_D(\mathcal{L} + D\mathcal{N}(\bar{f}))\backslash\{\lambda_0\}} Re(\lambda) \leq \sup_{\lambda \in \sigma_p(\mathcal{L} + D\mathcal{N}(\bar{f}))\backslash\{\lambda_0\}} Re(\lambda) < Re(\lambda_0)$$

if and only if

$$\xi(0) = \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right) - 1 > 0\,.$$

We can now summarize the results of this section in the following criterion.

Criterion. *(Instability) The $\alpha-$growth bound of the operator $\mathcal{L}$ can be found explicitly, i.e., $\omega_1(\mathcal{L}) = -\infty$. Furthermore, if*

$$\xi(0) = \int_{x_0}^{x_1} \frac{q(x)}{g(x)} \exp\left(-\int_{x_0}^{x} \frac{w(s)}{g(s)}\, ds\right) - 1 > 0\,,$$

then the zero solution of the semilinear evolution equation (3) *is unstable in the sense described in Proposition 9.*

4. Concluding remarks

Expressing stability results in terms of characteristic functions has been popular since A. Lotka published his pioneering article on population modeling [17]. The characteristic function, $\xi(\lambda)$, that we derived in this paper is easy to compute. For example, consider a hypothetical scenario for which the model parameters are given by $x_0 = 1$, $x_1 = 1000$, $q(x) = \ln(x)$, $g(x) = \frac{1}{10}x(1001 - x)$, $w(x) = \frac{1}{1000}(x - 1)^{1.17}$. In this case the zero solution of the evolution equation (1) is locally asymptotically stable. On the other hand, decreasing the growth rate twofold gives an unstable zero solution. This might not be very intuitive at first glance. However, when we decrease the growth rate twofold, aggregates will grow slower, but new cells keep entering single cell population at the same rate (fecundity rate $q(x)$). Thus, to keep the zero solution stable we should also decrease the fecundity rate (at least) twofold. Furthermore, as one might expect, doubling the fecundity rate $q(x)$ also makes the zero solution unstable.

As a future research, we plan to extend the results of this paper to nontrivial solutions (stationary, self-similar, etc.) of this aggregation-growth model as well as apply this linearization method to a size-specific aggregation-fragmentation model considered in [6].

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References

References

[1] Ackleh, A. S. (1997). Parameter estimation in a structured algal coagulation-fragmentation model. *Nonlinear Analysis*, 28(5):837–854.

[2] Ackleh, A. S. and Fitzpatrick, B. G. (1997). Modeling aggregation and growth processes in an algal population model: analysis and computations. *Journal of Mathematical Biology*, 35(4):480–502.

[3] Adams, R. and Fournier, J. (2003). *Sobolev spaces*. Elsevier Ltd, Oxford, UK.

[4] Banasiak, J. and Lamb, W. (2009). Coagulation, fragmentation and growth processes in a size structured population. *Discrete and Continuous Dynamical Systems - Series B*, 11(3):563–585.

[5] Banks, H. T. and Kappel, F. (1989). Transformation semigroups and L1-approximation for size structured population models. *Semigroup Forum*, 38(1):141–155.

[6] Bortz, D. M., Jackson, T. L., Taylor, K. A., Thompson, A. P., and Younger, J. G. (2008). {\it Klebsiella pneumoniae} flocculation dynamics. *Bull. Math. Biology*, 70(3):745–68.

[7] Engel, K. J. and Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Math.* Springer-Verlag, New York, NY.

[8] Falkowski, P. G. (1994). The role of phytoplankton photosynthesis in global biogeochemical cycles. *Photosynthesis research*, 39(3):235–58.

[9] Farkas, J. Z. and Hagen, T. (2007). Stability and regularity results for a size-structured population model. *Journal of Mathematical Analysis and Applications*, 328(1):119–136.

[10] Greiner, G. and Nagel, R. (1988). Growth of cell populations via one-parameter semigroups of positive operators. *Mathematics applied to science*, 13(2):934937.

[11] Griffin, S. L. and Rippingale, R. J. (2001). Zooplankton grazing dynamics: top-down control of phytoplankton and its relationship to an estuarine habitat. *Hydrological Processes*, 15(13):2453–2464.

[12] Hanche-Olsen, H. and Holden, H. (2010). The Kolmogorov-Riesz compactness theorem. *Expositiones Mathematicae*, 28(4):385–394.

[13] Hunter, J. K. and Nachtergaele, B. (2001). *Applied analysis*. World Scientific Publishing Company, Singapore, 2 edition.

[14] Kato, N. (1995). A principle of linearized stability for nonlinear evolution equations. *Transactions of the American Mathematical Society*, 347(8):2851–2868.

[15] Kuratowski, K. (1966). *Topology*. Academic Press, New York, NY, USA.

[16] Lalli, C. and Parsons, T. (1993). *Biological Oceanography: An Introduction*. Elsevier Butterworth-Heinemann, Oxford, UK, 2 edition.

[17] Lotka, A. (1922). The stability of the normal age distribution. *Proceedings of the National Academy of Sciences of the United States of America*, 8(11):339–345.

[18] Nagel, R., editor (1986). *One-parameter Semigroups of Positive Operators*, volume 1184 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg.

[19] Reynolds, C. (1984). *The ecology of freshwater phytoplankton*. Cambridge University Press, New York, NY, USA.

[20] Webb, G. F. (1985). *Theory of Nonlinear Age-Dependent Population Dynamics*. Number 89 in Pure and Applied Mathematics. Marcel Dekker, New York, NY.

[21] Weis, L. (1995). The stability of positive semigroups on Lp spaces. *Proceedings of the American Mathematical Society*, 123(10):3089–3094.